

# A FUNCTION DETERMINED BY A HYPERSURFACE OF POSITIVE CHARACTERISTIC

KOSUKE OHTA (MEIJI UNIVERSITY)

**ABSTRACT.** Let  $R = k[[X_1, \dots, X_{n+1}]]$  be a formal power series ring over a perfect field  $k$  of prime characteristic  $p > 0$ , and let  $\mathfrak{m} = (X_1, \dots, X_{n+1})$  be the maximal ideal of  $R$ . Suppose  $0 \neq f \in \mathfrak{m}$ . In this paper, we introduce a function  $\xi_f(x)$  associated with a hypersurface defined on the closed interval  $[0, 1]$  in  $\mathbb{R}$ . The Hilbert-Kunz function and the F-signature of a hypersurface appear as the values of our function  $\xi_f(x)$  on the interval's endpoints. The F-signature of the pair, denoted by  $s(R, f^t)$ , was defined in [3]. Our function  $\xi_f(x)$  is integrable, and the integral  $\int_t^1 \xi_f(x) dx$  is just  $s(R, f^t)$  for any  $t \in [0, 1]$ .

## 1. INTRODUCTION

In this paper, we introduce a function  $\xi_f(x)$  associated with a hypersurface defined on the closed interval  $[0, 1]$  in  $\mathbb{R}$ . The Hilbert-Kunz function and the F-signature of a hypersurface appear as the values of our function  $\xi_f(x)$  on the interval's endpoints.

**Definition 1.1.** Let  $(R, \mathfrak{n}, k)$  be a  $d$ -dimensional Noetherian local ring of prime characteristic  $p > 0$ . The *Hilbert-Kunz multiplicity* of  $R$  is

$$e_{HK}(R) = \lim_{e \rightarrow \infty} \frac{\ell(R/\mathfrak{n}^{[p^e]})}{p^{ed}},$$

where  $\ell(R/\mathfrak{n}^{[p^e]})$  is the length of  $R/\mathfrak{n}^{[p^e]}$ , and  $\mathfrak{n}^{[p^e]}$  is the ideal generated by all the  $p^e$ -th powers of elements of  $\mathfrak{n}$ . Monsky showed that this limit always exists ([8]).

Let  $(R, \mathfrak{n}, k)$  be a  $d$ -dimensional Noetherian local ring of prime characteristic  $p > 0$ , and let  $F : R \rightarrow R$  be the Frobenius map, that is  $F(x) = x^p$  for any  $x \in R$ . Taking a positive integer  $e > 0$ , we obtain the ring  $R$ , denoted by  $F_*^e R$ , viewed as an  $R$ -module via the  $e$ -th Frobenius map. The element  $s$  in  $F_*^e R$  is sometimes denoted by  $F_*^e(s)$ . For  $a, e \in R$ , we have  $a \cdot F_*^e(s) = F_*^e(a^{p^e} s)$ . We define the F-signature as follows.

**Definition 1.2.** Let  $(R, \mathfrak{n}, k)$  be a  $d$ -dimensional complete Cohen-Macaulay local ring of prime characteristic  $p > 0$ . Assume that  $R$  is an F-finite ring, namely, the Frobenius map  $F : R \rightarrow R$  is finite. Suppose  $F_*^e R \simeq R^{\oplus a_e} \oplus M_e$  with some integer  $a_e$  and some  $R$ -module  $M_e$ , where  $M_e$  has no free direct summands. The number  $a_e$  is called  *$e$ -th Frobenius splitting number* of  $R$ . Then,

$$s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$$

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is called the *F-signature* of  $R$ . Tucker showed this limit always exists ([9], Theorem 4.9).

In the rest of this paper, let  $n \geq 1$ . Let  $R = k[[X_1, \dots, X_{n+1}]]$  be a formal power series ring over a perfect field  $k$  of prime characteristic  $p > 0$ , and let  $\mathfrak{m} = (X_1, \dots, X_{n+1})$  be the maximal ideal of  $R$ . Suppose  $0 \neq f \in \mathfrak{m}$ . Rings of the form  $R/(f)$  are called “*n-dimensional hypersurface*”.

**Definition 1.3.** We define

$$M_{e,t} = \frac{(f^t) + \mathfrak{m}^{[p^e]}}{(f^{t+1}) + \mathfrak{m}^{[p^e]}} \simeq \frac{R}{[(f^{t+1}) + \mathfrak{m}^{[p^e]}] : f^t},$$

where  $e \geq 0$  and  $t \geq 0$  are integers.

Then we have, for any  $t \geq 0$ , the surjection

$$M_{e,t} \rightarrow M_{e,t+1}$$

because  $[(f^{t+1}) + \mathfrak{m}^{[p^e]}] : f^t \subset [(f^{t+2}) + \mathfrak{m}^{[p^e]}] : f^{t+1} \subset R$ .

Let  $\overline{R} = R/\mathfrak{m}^{[p^e]}$ . Then, remark that  $M_{e,t} = f^t \overline{R} / f^{t+1} \overline{R}$ .

**Definition 1.4.** We define

$$C_{e,t} = \frac{\ell_R(M_{e,t})}{p^{en}},$$

where  $\ell_R(M_{e,t})$  is the length as an  $R$ -module.

Then we have

$$p^e \geq C_{e,0} \geq C_{e,1} \geq C_{e,2} \geq \dots \geq C_{e,p^e-1} \geq C_{e,p^e} = C_{e,p^e+1} = \dots = 0. \quad (1)$$

A sequence of functions  $\{\xi_{f,e} : [0, 1] \rightarrow \mathbb{R}\}_{e \geq 0}$  is defined by

$$\xi_{f,e}(x) = \begin{cases} C_{e, \lfloor xp^e \rfloor} & (0 \leq x < 1) \\ C_{e, p^e-1} & (x = 1) \end{cases},$$

where  $\lfloor xp^e \rfloor = \max \{a \in \mathbb{Z} | xp^e \geq a\}$  is the floor function. By the definition, we have  $\int_0^1 \xi_{f,e}(x) dx = 1$  because

$$\begin{aligned} \int_0^1 \xi_{f,e}(x) dx &= \frac{1}{p^e} (C_{e,0} + C_{e,1} + C_{e,2} + \dots + C_{e,p^e-1}) \\ &= \frac{1}{p^e} \times \frac{1}{p^{en}} (\ell_R(M_{e,0}) + \ell_R(M_{e,1}) + \dots + \ell_R(M_{e,p^e-1})) \\ &= \frac{1}{p^{e(n+1)}} \ell_R(R/\mathfrak{m}^{[p^e]}) \\ &= \frac{1}{p^{e(n+1)}} \times p^{e(n+1)} \\ &= 1. \end{aligned}$$

**Definition 1.5.** We define the function  $\xi_f(x)$  by

$$\xi_f(x) = \limsup_{e \rightarrow \infty} \xi_{f,e}(x)$$

for  $x \in [0, 1]$ .

By the inequalities (1),  $\xi_f(x)$  is decreasing on  $[0, 1]$ . If  $\lim_{e \rightarrow \infty} \xi_{f,e}(\alpha)$  exists, then  $\xi_f(\alpha) = \lim_{e \rightarrow \infty} \xi_{f,e}(\alpha)$ . The sequence  $\{C_{e,0}\}_e$  is increasing by Lemma 2.1 in section 2.

$$\lim_{e \rightarrow \infty} C_{e,0} = \lim_{e \rightarrow \infty} \frac{\ell_R(M_{e,0})}{p^{en}} = \lim_{e \rightarrow \infty} \frac{\ell_R(R/(f) + \mathfrak{m}^{[p^e]})}{p^{en}}.$$

This limit exists and is called the Hilbert-Kunz multiplicity of  $R/(f)$ , denoted by  $e_{HK}(R/(f))$ . Therefore, by (1),  $\limsup_{e \rightarrow \infty} \xi_{f,e}(\alpha)$  is not  $+\infty$  for any  $\alpha \in [0, 1]$ . We shall give an example that  $\lim_{e \rightarrow \infty} \xi_{f,e}(\alpha)$  does not exist for some  $f \in R$  and  $\alpha \in [0, 1]$  in section 3. We have

$$\xi_f(0) = e_{HK}(R/(f)).$$

Therefore,  $\xi_f(x)$  is a bounded and decreasing function on  $[0, 1]$ . In particular,  $\xi_f(x)$  is integrable, and has at most countably many points of discontinuity on  $[0, 1]$ .

The main theorem of this paper is the following:

- Theorem 1.6.** 1) The function  $\xi_f(x)$  is decreasing. There exists a countable subset  $C$  of the interval  $[0, 1]$  such that  $\xi_f(x)$  is continuous at any  $\alpha \in [0, 1] - C$ . Moreover,  $\xi_f(x)$  is continuous at 0 and 1.
- 2) If  $\xi_f(x)$  is continuous at  $\alpha \in [0, 1]$ , then  $\lim_{e \rightarrow \infty} \xi_{f,e}(\alpha) = \xi_f(\alpha)$ .
- 3) We have  $\xi_f(0) = e_{HK}(R/(f))$ , and also  $\xi_f(1) = s(R/(f))$ .
- 4) Suppose that  $\xi_f(1) = 0$ , then  $\text{fpt}(f) = \inf\{\alpha \in [0, 1] \mid \xi_f(\alpha) = 0\}$  holds, where  $\text{fpt}(f) = \lim_{e \rightarrow \infty} \frac{\mu_f(p^e)}{p^e}$  is the F-pure threshold of  $f$ , where  $\mu_f(p^e) = \min\{t \geq 1 \mid f^t \in \mathfrak{m}^{[p^e]}\}$ .
- 5) The function  $\xi_f(x)$  is integrable, and we have  $\int_{\frac{a}{p^e}}^{\frac{a+1}{p^e}} \xi_f(x) dx = \frac{\ell_R(M_{e,a})}{p^{e(n+1)}}$  for integers  $0 \leq a < p^e$ . In particular,  $\int_0^1 \xi_f(x) dx = 1$  holds.
- 6) If  $R/(f)$  is normal then  $\xi'_f(0) = 0$ , where  $\xi'_f$  is the derivative of  $\xi_f$ .

**Remark 1.7.** By Theorem 1.1 and Proposition 3.2 (i) in [2], we know that above  $\text{fpt}(f)$  is a positive rational number. Note that F-pure thresholds are defined as the smallest of the F-jumping exponents in [2].

**Remark 1.8.** We define the function  $\varphi_f(x)$  on  $[0, 1]$  as follows;

$$\varphi_f(x) = \int_0^x \xi_f(t) dt.$$

Actually, we have

$$\varphi_f(x) = \lim_{e \rightarrow \infty} \frac{1}{p^e} (C_{e,0} + C_{e,1} + \cdots + C_{e, \lfloor xp^e \rfloor - 1}).$$

Since  $\xi_f(x)$  is bounded and integrable on  $[0, 1]$ ,  $\varphi_f(x)$  is Lipschitz continuous on  $[0, 1]$ . In particular,  $\varphi_f(x)$  is continuous on  $[0, 1]$ . We can rewrite 3) and 4) in Theorem 1.6 as follows;

- 3') The function  $\varphi_f(x)$  is differentiable at  $x = 0$  and 1, and  $\varphi'_f(0) = e_{HK}(R/(f))$  and  $\varphi'_f(1) = s(R/(f))$ .

4') Suppose that  $s(R/(f)) = 0$ , then

$$\text{fpt}(f) = \inf\{\alpha \in [0, 1] \mid \varphi_f(\alpha) = 1\}$$

holds.

Let  $(A, \mathfrak{n})$  be an F-finite regular local ring, and let  $g \in A$  be a non-zero element. In [3], the F-signature of the pair  $(A, g^t)$  for any real number  $t \in [0, 1]$  is denoted as

$$s(A, g^t) = \lim_{e \rightarrow \infty} \frac{1}{p^{e(n+1)}} \ell_A \left( \frac{A}{\mathfrak{n}^{[p^e]} : g^{\lceil t(p^e-1) \rceil}} \right).$$

Using 5) in Theorem 1.6, we know

$$1 - \varphi_f(x) = \int_t^1 \xi_f(x) dx = s(R, f^t)$$

for  $t \in [0, 1]$ . Moreover, if we know that  $\xi_f(x)$  is continuous at 0 and 1 (see Theorem 1.6 1)), we obtain 3) in Theorem 1.6 immediately from Theorem 4.4 in [3].

In section 2, we shall prove Theorem 1.6. The following corollary immediately follows from Theorem 1.6 3) and 5)

**Corollary 1.9.**  $e_{HK}(R/(f)) \times \text{fpt}(f) \geq 1$ .

**Example 1.10.** Suppose  $R = k[[X_1, X_2, \dots, X_{n+1}]]$  and  $\alpha > 0$ . Then  $e_{HK}(R/(X_1^\alpha)) = \alpha$  and  $\text{fpt}(X_1^\alpha) = \frac{1}{\alpha}$ . Therefore, if  $\tau(f) = X_1^\alpha$  for a linear transformation  $\tau$  (for example,  $f = X_1 + X_2$ ), then  $e_{HK}(R/(f)) \times \text{fpt}(f) = 1$  and  $s(R/(f)) = 1$  (see section 3). We do not know another example that the equality holds in Corollary 1.9.

**Remark 1.11.** By Theorem 1.6 1), 3) and 5), we immediately know that  $e_{HK}(R/(f)) = 1$  if and only if  $s(R/(f)) = 1$ . These conditions are equivalent to that  $R/(f)$  is regular by the following results.

- 1) *Let  $S$  be an unmixed local ring of positive characteristic. Then  $e_{HK}(S) = 1$  if and only if  $S$  is regular ([10], Theorem 1.5).*
- 2) *Let  $S$  be a reduced F-finite Cohen-Macaulay local ring of positive characteristic. Then  $s(S) = 1$  if and only if  $S$  is regular ([5], Corollary 16).*

**Remark 1.12.** Let  $m < n = \dim R/(f)$ , and set  $a_e = \ell(M_{e, p^e-1})$ . Assume that  $a_e = \alpha p^{em} + o(p^{em})$ , that is  $\lim_{e \rightarrow \infty} \frac{a_e}{p^{em}} = \alpha$ . Let  $g_e = a_e - \alpha p^{em}$ . Then

$$\begin{aligned} \varphi_f(1) - \varphi_f\left(\frac{p^e - 1}{p^e}\right) &= \sum_{i=0}^{p^e-1} \frac{\ell(M_{e, i})}{p^{e(n+1)}} - \sum_{i=0}^{p^e-2} \frac{\ell(M_{e, i})}{p^{e(n+1)}} \\ &= \frac{\ell(M_{e, p^e-1})}{p^{e(n+1)}} \\ &= \frac{\alpha}{p^{e(n-m+1)}} + \frac{g_e}{p^{e(n+1)}} \end{aligned}$$

holds. Let  $x = \frac{p^e - 1}{p^e}$ . Since  $x - 1 = -\frac{1}{p^e}$ , we know

$$\varphi_f(x) = \varphi_f(1) + (-1)^{n-m} \alpha (x - 1)^{n-m+1} + o((x - 1)^{n-m+1}). \quad (2)$$

Since  $\varphi_f(x)$  is continuous on  $[0, 1]$  from Remark 1.7,  $\varphi_f(x)$  has the form of (2) around the point  $x = 1$ . Therefore, if  $\varphi_f(x)$  is equal to its Taylor series around the point  $x = 1$ , we obtain that

$$\begin{aligned}\varphi_f^{(i)}(1) &= \begin{cases} 0 & (i = 1, 2, \dots, n-m) \\ (-1)^{n-m}(n-m+1)!\alpha & (i = n-m+1) \end{cases}, \\ \xi_f^{(i)}(x) &= \begin{cases} 0 & (i = 1, 2, \dots, n-m-1) \\ (-1)^{n-m}(n-m+1)!\alpha & (i = n-m) \end{cases}.\end{aligned}$$

## 2. PROOF OF MAIN THEOREM

Let  $F : R \rightarrow R$  be the Frobenius map  $a \mapsto a^p$ . Since  $k$  is perfect, we have  $F_*R \simeq R^{\oplus p^{n+1}}$ , where  $F_*R$  stands for  $F_*^1 R$ . Therefore,  $\frac{(f^{pt}) + \mathfrak{m}^{[p^{e+1}]}}{(f^{pt+p}) + \mathfrak{m}^{[p^{e+1}]}} = M_{e,t} \otimes_R F_*R \simeq (M_{e,t})^{\oplus p^{n+1}}$  for all  $e, t \geq 0$ . Consequently,

$$p \times C_{e,t} = C_{e+1,pt} + C_{e+1,pt+1} + \dots + C_{e+1,pt+p-1}, \quad (3)$$

where the sum on the right-hand side of (3) has  $p$ -terms. That is,  $C_{e,t}$  is the mean of  $C_{e+1,pt}, C_{e+1,pt+1}, \dots, C_{e+1,pt+p-1}$ . Therefore, by (1) and (3), we obtain the following inequalities immediately.

**Lemma 2.1.**  $C_{e+1,pt} \geq C_{e,t} \geq C_{e+1,pt+p-1}$ .

Hence, by (1) and Lemma 2.1, we have

$$\begin{array}{ccc} C_{e, \lfloor xp^e \rfloor - 1} & \underset{\text{by Lemma 2.1}}{\geq} & C_{e+1, (\lfloor xp^e \rfloor - 1)p + (p-1)} \geq C_{e+1, \lfloor xp^{e+1} \rfloor - 1} \\ \vee & & \vee \\ C_{e, \lfloor xp^e \rfloor} & & C_{e+1, \lfloor xp^{e+1} \rfloor} \\ \vee & & \vee \\ C_{e, \lceil xp^e \rceil} & \underset{\text{by Lemma 2.1}}{\leq} & C_{e+1, \lceil xp^e \rceil p} \leq C_{e+1, \lceil xp^{e+1} \rceil} \end{array}$$

and here, we note that  $\lfloor xp^e \rfloor p \leq \lfloor xp^{e+1} \rfloor$  and  $\lceil xp^e \rceil p \geq \lceil xp^{e+1} \rceil$ . Therefore, the sequence  $\{C_{e, \lfloor xp^e \rfloor - 1}\}_e$  is decreasing, the sequence  $\{C_{e, \lceil xp^e \rceil}\}_e$  is increasing, and  $C_{e, \lfloor xp^e \rfloor - 1} \geq C_{e, \lceil xp^e \rceil}$  for all  $e \geq 0$  by the inequalities (1). Consequently, the limits  $\lim_{e \rightarrow \infty} C_{e, \lfloor xp^e \rfloor - 1}$  and  $\lim_{e \rightarrow \infty} C_{e, \lceil xp^e \rceil}$  exist in  $\mathbb{R}$ . In particular,

$$C_{e, \lfloor \alpha p^e \rfloor - 1} \geq \lim_{e \rightarrow \infty} C_{e, \lfloor \alpha p^e \rfloor - 1} \geq \xi_f(\alpha) \geq \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil} \geq C_{e, \lceil \alpha p^e \rceil} \geq 0 \quad (4)$$

holds for any  $\alpha \in (0, 1]$  and  $e$  satisfying  $\lfloor \alpha p^e \rfloor - 1 \geq 0$ .

**Lemma 2.2.** We set  $\overline{C}(\alpha) = \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil}$  for  $\alpha \in [0, 1]$  and  $\underline{C}(\beta) = \lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor - 1}$  for  $\beta \in (0, 1]$ .

1) For  $\alpha \in [0, 1]$  and any integer  $i \geq 0$ ,  $\{C_{e+1, \lceil \alpha p^e \rceil p + i}\}_e$  is an increasing sequence. The limits  $\lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p + i}$  and  $\lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil + k}$  exist for any non-negative integers  $i, k \geq 0$ . Furthermore,

$$\overline{C}(\alpha) = \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p + i} = \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil + k} \quad (5)$$

holds.

- 2) For  $\beta \in (0, 1]$  and any integer  $i > 0$ ,  $\{C_{e+1, \lfloor \beta p^e \rfloor p-i}\}_e$  is a decreasing sequence. The limits  $\lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-i}$  and  $\lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor -k}$  exist for any positive integers  $i, k > 0$ . Furthermore,

$$\underline{C}(\beta) = \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-i} = \lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor -k} \quad (6)$$

holds.

*Proof.* Let  $\alpha \in [0, 1]$  and  $\beta \in (0, 1]$ , and let  $k \geq 0$  and  $\ell > 0$  be integers. We know

$$\begin{cases} (\lceil \alpha p^e \rceil p + k)p = \lceil \alpha p^e \rceil p^2 + kp \geq \lceil \alpha p^{e+1} \rceil p + kp \geq \lceil \alpha p^{e+1} \rceil p + k \\ (\lfloor \beta p^e \rfloor p - \ell)p + (p-1) \leq \lfloor \beta p^e \rfloor p^2 - \ell p + (p-1)\ell \leq \lfloor \beta p^{e+1} \rfloor p - \ell \end{cases},$$

and therefore

$$\begin{cases} C_{e+1, \lceil \alpha p^e \rceil p+k} \leq C_{e+2, (\lceil \alpha p^e \rceil p+k)p} \leq C_{e+2, \lceil \alpha p^{e+1} \rceil p+k} \leq \lim_{e \rightarrow \infty} C_{e, 0} \\ C_{e+1, \lfloor \beta p^e \rfloor p-\ell} \geq C_{e+2, (\lfloor \beta p^e \rfloor p-\ell)p+(p-1)} \geq C_{e+2, \lfloor \beta p^{e+1} \rfloor p-\ell} \geq 0 \end{cases}$$

by (1) and Lemma 2.1. Hence,  $\{C_{e+1, \lceil \alpha p^e \rceil p+k}\}_e$  is increasing and bounded.  $\{C_{e+1, \lfloor \beta p^e \rfloor p-\ell}\}_e$  is decreasing and bounded. Therefore,  $\lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p+k}$  and  $\lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-\ell}$  exist.

Next, we shall show that

$$\overline{C}(\alpha) = \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p+i} \quad (7)$$

$$\underline{C}(\beta) = \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-j} \quad (8)$$

holds for any integers  $0 \leq i \leq p-1$  and  $1 \leq j \leq p$ . We have

$$\begin{cases} p \times C_{e, \lceil \alpha p^e \rceil} = C_{e+1, \lceil \alpha p^e \rceil p} + C_{e+1, \lceil \alpha p^e \rceil p+1} + \cdots + C_{e+1, \lceil \alpha p^e \rceil p+p-1} \\ p \times C_{e, \lfloor \beta p^e \rfloor -1} = C_{e+1, \lfloor \beta p^e \rfloor p-p} + C_{e+1, \lfloor \beta p^e \rfloor p-(p-1)} + \cdots + C_{e+1, \lfloor \beta p^e \rfloor p-1} \end{cases}$$

by (3). Thus, it holds that

$$\begin{cases} p \times \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil} = \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p} + \cdots + \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p+p-1} \\ p \times \lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor -1} = \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-p} + \cdots + \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-1} \end{cases}.$$

On the other hand, we have

$$\begin{cases} \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil} = \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p} \geq \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p+1} \geq \cdots \geq \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p+p-1} \\ \lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor -1} = \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-1} \leq \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-2} \leq \cdots \leq \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p-p} \end{cases}$$

since  $C_{e, \lceil \alpha p^e \rceil} \leq C_{e+1, \lceil \alpha p^e \rceil p} \leq C_{e+1, \lceil \alpha p^{e+1} \rceil}$  and  $C_{e, \lfloor \beta p^e \rfloor -1} \geq C_{e+1, \lfloor \beta p^e \rfloor p-1} \geq C_{e+1, \lfloor \beta p^{e+1} \rfloor -1}$ . Consequently, we have the equations (7) and (8).

In order to complete the proof of the assertion 1), we have the inequalities

$$\begin{aligned} C_{e, \lceil \alpha p^e \rceil +k} &\leq C_{e+1, (\lceil \alpha p^e \rceil +k)p} \\ &= C_{e+1, \lceil \alpha p^e \rceil p+kp} \\ &\leq C_{e+1, \lceil \alpha p^e \rceil p+k} \\ &\leq C_{e+1, \lceil \alpha p^{e+1} \rceil +k} \end{aligned}$$

for any  $k \geq 1$ . Hence,

$$\lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil +k} = \lim_{e \rightarrow \infty} C_{e+1, \lceil \alpha p^e \rceil p+k}$$

holds. Therefore, we obtain the equation (5).

In order to complete the proof of the assertion 2), we have the inequalities

$$\begin{aligned} C_{e, \lfloor \beta p^e \rfloor - k} &\geq C_{e+1, (\lfloor \beta p^e \rfloor - k)p + p - 1} \\ &= C_{e+1, \lfloor \beta p^e \rfloor p - (k-1)p - 1} \\ &\geq C_{e+1, \lfloor \beta p^e \rfloor p - k} \\ &\geq C_{e+1, \lfloor \beta p^{e+1} \rfloor - k} \end{aligned}$$

for any  $k \geq 2$ . Hence,

$$\lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor - k} = \lim_{e \rightarrow \infty} C_{e+1, \lfloor \beta p^e \rfloor p - k}$$

holds. Therefore, we obtain the equation (6).  $\square$

**Proposition 2.3.** 1) For  $\alpha \in [0, 1)$ ,  $\lim_{x \rightarrow \alpha+0} \xi_f(x) = \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil}$  holds.  
2) For  $\beta \in (0, 1]$ ,  $\lim_{x \rightarrow \beta-0} \xi_f(x) = \lim_{e \rightarrow \infty} C_{e, \lfloor \beta p^e \rfloor - 1}$  holds.

In particular, we have

$$\begin{cases} \lim_{x \rightarrow +0} \xi_f(x) = \lim_{e \rightarrow \infty} C_{e, 0} = \xi_f(0) \\ \lim_{x \rightarrow 1-0} \xi_f(x) = \lim_{e \rightarrow \infty} C_{e, p^e - 1} = \xi_f(1) \end{cases},$$

that is to say that  $\xi_f(x)$  is continuous at  $x = 0$  and  $1$ .

*Proof.* 1) First, we show  $\lim_{x \rightarrow \alpha+0} \xi_f(x) \leq \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil}$ . Take  $x_0 > \alpha$ . For a large enough number  $e'$ , we may assume that  $\alpha p^{e'} \leq x_0 p^{e'} - 2$  holds. Then,  $\lceil \alpha p^{e'} \rceil \leq \lfloor x_0 p^{e'} \rfloor - 1$ . Hence, by the inequalities (1) and (4),

$$\xi_f(x_0) \leq C_{e', \lfloor x_0 p^{e'} \rfloor - 1} \leq C_{e', \lceil \alpha p^{e'} \rceil} \leq \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil}$$

as desired.

Next, we shall show the opposite inequality. By Lemma 2.2 1), we have only to show that

$$\lim_{x \rightarrow \alpha+0} \xi_f(x) \geq \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil + 1}.$$

For any  $e \geq 0$ ,  $\alpha < \frac{\lceil \alpha p^e \rceil + 1}{p^e}$ . Hence, there exists a real number  $x_1 \in \mathbb{R}$  such that

$$\alpha < x_1 < \frac{\lceil \alpha p^e \rceil + 1}{p^e}. \text{ Then } \lceil x_1 p^e \rceil \leq \lceil \alpha p^e \rceil + 1, \text{ and therefore}$$

$$\lim_{x \rightarrow \alpha+0} \xi_f(x) \geq \xi_f(x_1) \geq C_{e, \lceil x_1 p^e \rceil} \geq C_{e, \lceil \alpha p^e \rceil + 1}$$

for any  $e \geq 0$  because we have the inequalities (1) and (4), and  $\xi_f(x)$  is decreasing. Consequently,

$$\lim_{x \rightarrow \alpha+0} \xi_f(x) \geq \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil + 1}$$

as desired.

2) It is proved in the same way as 1).  $\square$

**Remark 2.4.** From the inequalities (1), we have

$$C_{e, \lfloor \alpha p^e \rfloor - 1} \geq \xi_{f,e}(\alpha) = C_{e, \lfloor \alpha p^e \rfloor} \geq C_{e, \lceil \alpha p^e \rceil}$$

for any  $\alpha \in [0, 1]$ . Hence, if

$$\lim_{e \rightarrow \infty} C_{e, \lfloor \alpha p^e \rfloor - 1} = \lim_{e \rightarrow \infty} C_{e, \lceil \alpha p^e \rceil},$$

there exists  $\lim_{e \rightarrow \infty} \xi_{f,e}(\alpha)$  in  $\mathbb{R}$ , and it is equal to  $\xi_f(\alpha)$ .

**Corollary 2.5.** If  $\xi_f(x)$  is continuous at  $\alpha \in [0, 1]$  then  $\lim_{e \rightarrow \infty} \xi_{f,e}(\alpha)$  exists, so that it is equal to  $\xi_f(\alpha)$ .

*Proof.* The proof is obtained from Remark 2.4 immediately.  $\square$

We have just shown Theorem 1.6 1).

We obtain the following Corollary 2.6 immediately from Proposition 2.3.

**Corollary 2.6.** We define  $\varphi_f(x)$  by

$$\varphi_f(x) = \int_0^x \xi_f(t) dt$$

for  $x \in [0, 1]$ . Then we have the followings.

- 1)  $\varphi_f(x)$  is differentiable at 0, and  $\varphi'_f(0) = \xi_f(0) = \lim_{e \rightarrow \infty} C_{e,0} = e_{HK}(R/(f))$ .
- 2)  $\varphi_f(x)$  is differentiable at 1, and  $\varphi'_f(1) = \xi_f(1) = \lim_{e \rightarrow \infty} C_{e,p^e-1}$ .

Set  $\mu_f(p^e) = \min\{t \geq 0 \mid f^t \in \mathfrak{m}^{[p^e]}\}$  for each  $e \geq 0$ . Since  $f^{\mu_f(p^e)} \in \mathfrak{m}^{[p^e]}$ ,  $f^{\mu_f(p^e)p} \in \mathfrak{m}^{[p^{e+1}]}$ . Hence  $\mu_f(p^e)p \geq \mu_f(p^{e+1})$ , and so

$$1 \geq \frac{\mu_f(p^e)}{p^e} \geq \frac{\mu_f(p^{e+1})}{p^{e+1}} \geq 0.$$

Since  $\left\{ \frac{\mu_f(p^e)}{p^e} \right\}_{e \geq 0}$  is decreasing and bounded below, the limit  $\lim_{e \rightarrow \infty} \frac{\mu_f(p^e)}{p^e}$  exists in  $\mathbb{R}$ , and it is called the F-pure threshold of  $f$ , denoted by  $\text{fpt}(f)$ . It is easy to see that  $\text{fpt}(f) \in (0, 1]$ , and  $\text{fpt}(f) = 1$  if and only if  $\mu_f(p^e) = p^e$  for any  $e \geq 1$ .

**Lemma 2.7.**  $C_{e,t} = 0$  if and only if  $t \geq \mu_f(p^e)$ .

*Proof.* If  $M_{e,t} = 0$ , then  $M_{e,t} = M_{e,t+1} = M_{e,t+2} = \cdots = M_{e,p^e} = 0$ . Hence,  $f^t \in \mathfrak{m}^{[p^e]}$ , and so  $t \geq \mu_f(p^e)$ . Conversely if  $t \geq \mu_f(p^e)$ , then  $f^t \in \mathfrak{m}^{[p^e]}$  holds.  $\square$

We start to prove Theorem 1.6. The assertion 1) follows from Proposition 2.3. The assertion 2) follows from Corollary 2.5. The first half of 3) follows from the definition of  $C_{e,0}$ . Now, we shall show 4).

*Proof.* First, we check that

$$\inf\{\alpha \in [0, 1] \mid \xi_f(\alpha) = 0\} \leq \text{fpt}(f).$$

If  $\text{fpt}(f) = 1$ , then the assertion is easy. Assume  $\text{fpt}(f) < 1$ . Let  $1 > \alpha > \text{fpt}(f)$ .

Since  $\text{fpt}(f) = \inf_{e \geq 0} \left\{ \frac{\mu_f(p^e)}{p^e} \right\}$ ,

$$\text{fpt}(f) \leq \frac{\mu_f(p^{e_1})}{p^{e_1}} < \alpha$$

holds for  $e_1 \gg 0$ . Then, it holds that

$$\begin{aligned} \xi_f(\alpha) &\leq \xi_f\left(\frac{\mu_f(p^{e_1})}{p^{e_1}}\right) \\ &= \limsup_{e \rightarrow \infty} C_{e, \lfloor \frac{\mu_f(p^{e_1})}{p^{e_1}} p^e \rfloor} \\ &= 0 \end{aligned}$$



because, by Lemma 2.7,

$$C_{e_1+s, \mu_f(p^{e_1})p^s} \leq C_{e_1+s, \mu_f(p^{e_1+s})} = 0$$

for any integers  $s \geq 0$ . Therefore,  $\xi_f(\alpha) = 0$  for all  $\alpha > \text{fpt}(f)$ , as desired. Conversely, suppose  $\alpha < \text{fpt}(f)$ . Hence, we have  $(\text{fpt}(f) - \alpha)p^{e'} \geq 1$  for  $e' \gg 0$ , and therefore  $\alpha p^{e'} \leq \text{fpt}(f)p^{e'} - 1$ . Then, since we have

$$\alpha \leq \frac{\text{fpt}(f)p^{e'} - 1}{p^{e'}} < \frac{\text{fpt}(f)p^{e'}}{p^{e'}} = \text{fpt}(f) \leq \frac{\mu_f(p^{e'})}{p^{e'}},$$

we obtain

$$\alpha \leq \frac{\mu_f(p^{e'}) - 1}{p^{e'}}.$$

Therefore,

$$\xi_f(\alpha) \geq \xi_f\left(\frac{\mu_f(p^{e'}) - 1}{p^{e'}}\right) \underset{\text{by (4)}}{\geq} \lim_{e \rightarrow \infty} C_{e, \lceil \frac{\mu_f(p^{e'}) - 1}{p^{e'}} p^e \rceil}$$

holds. We have  $C_{e', \mu_f(p^{e'}) - 1} \neq 0$  by Lemma 2.7. Since  $\left\{ C_{e, \lceil \frac{\mu_f(p^{e'}) - 1}{p^{e'}} p^e \rceil} \right\}_{e \geq 0}$  is an increasing sequence, we obtain  $\lim_{e \rightarrow \infty} C_{e, \lceil \frac{\mu_f(p^{e'}) - 1}{p^{e'}} p^e \rceil} > 0$ . Therefore,  $\xi_f(\alpha) > 0$  for all  $\alpha$  such that  $\alpha < \text{fpt}(f)$ , as desired.  $\square$

Next, we shall show 5).

*Proof.* Let  $F = \left\{ \alpha \in \left[ \frac{a}{p^e}, \frac{a+1}{p^e} \right] \mid \alpha \text{ is a discontinuity for } \xi_f(x) \right\}$  and  $\Omega = \left[ \frac{a}{p^e}, \frac{a+1}{p^e} \right] - F$ . Recall that  $F$  is a countable set, and  $\lim_{e \rightarrow \infty} \xi_{f,e}(\alpha) = \xi_f(\alpha)$  for any  $\alpha \in \Omega$  by Theorem 1.6 1), 2). Then, we have

$$\begin{aligned} \int_{\frac{a}{p^e}}^{\frac{a+1}{p^e}} \xi_f(x) dx &= \int_{\Omega} \xi_f(x) dx \\ &= \int_{\Omega} \lim_{e \rightarrow \infty} \xi_{f,e}(x) dx \\ &= \lim_{e \rightarrow \infty} \int_{\Omega} \xi_{f,e}(x) dx \\ &= \lim_{e \rightarrow \infty} \int_{\frac{a}{p^e}}^{\frac{a+1}{p^e}} \xi_{f,e}(x) dx \\ &= \frac{1}{p^e} C_{e, a} \end{aligned}$$

by Lebesgue's dominated convergence theorem, as desired.  $\square$

We shall show 6).

*Proof.* Let  $g, h : \mathbb{N} \rightarrow \mathbb{R}$  be functions. If there exists a positive constant  $C$  such that  $|h(n)| \leq Cg(n)$  for  $n \gg 0$ , then we write  $h(n) = O(g(n))$ . If  $R/(f)$  is normal, then there exists  $\beta(R/(f)) \in \mathbb{R}$  such that

$$e_{HK}(R/(f))p^{ne} + \beta(R/(f))p^{(n-1)e} = \ell_R(M_{e,0}) + O(p^{(n-2)e})$$

by Huneke-McDermott-Monsky [6]. Since a hypersurface is Gorenstein, that  $\beta(R/(f)) = 0$  follows from Corollary 1.4 in Kurano [7]. Therefore, we have

$$e_{HK}(R/(f))p^{ne} = \ell_R(M_{e,0}) + O(p^{(n-2)e}). \quad (9)$$

First, we shall show that

$$\left| \frac{\xi_f\left(\frac{1}{p^s}\right) - \xi_f(0)}{\frac{1}{p^s}} \right| \longrightarrow 0 \quad (s \rightarrow \infty).$$

Since the sequence  $\{C_{s+i,p^i}\}_{i \geq 0}$  is increasing, we have

$$\xi_f\left(\frac{1}{p^s}\right) = \limsup_{e \rightarrow \infty} C_{e, \lfloor p^{e-s} \rfloor} \geq C_{s,1}.$$

Hence, we obtain

$$\left| \frac{\xi_f\left(\frac{1}{p^s}\right) - \xi_f(0)}{\frac{1}{p^s}} \right| = \frac{\xi_f(0) - \xi_f\left(\frac{1}{p^s}\right)}{\frac{1}{p^s}} \leq \frac{\xi_f(0) - C_{s,1}}{\frac{1}{p^s}}.$$

Set  $\lambda_i(e)$  as  $e_{HK}(R/(f))p^{en} - \ell_R(M_{e,i})$  for each  $e \geq 0$  and  $0 \leq i \leq p-1$ . Note that

$$0 \leq \lambda_0(e) \leq \lambda_1(e) \leq \cdots \leq \lambda_{p-1}(e).$$

Since we have,

$$p \times \frac{\ell_R(M_{s-1,0})}{p^{(s-1)n}} = \frac{\ell_R(M_{s,0})}{p^{sn}} + \frac{\ell_R(M_{s,1})}{p^{sn}} + \cdots + \frac{\ell_R(M_{s,p-1})}{p^{sn}}$$

for any  $s \geq 1$  by (3), then we obtain

$$p \times \frac{\lambda_0(s-1)}{p^{(s-1)n}} = \frac{\lambda_0(s)}{p^{sn}} + \frac{\lambda_1(s)}{p^{sn}} + \cdots + \frac{\lambda_{p-1}(s)}{p^{sn}}.$$

Hence, since

$$p \times \frac{\lambda_0(s-1)}{p^{(s-1)n}} \geq \frac{\lambda_1(s)}{p^{sn}},$$

it holds that

$$p^2 \times \frac{\lambda_0(s-1)}{p^{(s-1)(n-1)}} \geq \frac{\lambda_1(s)}{p^{s(n-1)}} \geq 0.$$

Therefore,

$$\begin{aligned} \frac{\xi_f(0) - C_{s,1}}{\frac{1}{p^s}} &= \frac{p^s}{p^{sn}} (e_{HK}(R/(f))p^{sn} - C_{s,1} \times p^{sn}) \\ &= \frac{\lambda_1(s)}{p^{s(n-1)}} \\ &\leq p^2 \times \frac{\lambda_0(s-1)}{p^{(s-1)(n-1)}} \\ &= \frac{p^2}{p^{s-1}} \times \frac{\lambda_0(s-1)}{p^{(s-1)(n-2)}} \\ &\rightarrow 0 \quad (e \rightarrow \infty) \end{aligned}$$

by the equation (9). Consequently, for any positive real number  $\varepsilon > 0$ , there exists a natural number  $s_0 \in \mathbb{N}$  such that  $s \geq s_0$  implies that

$$\left| \frac{\xi_f\left(\frac{1}{p^s}\right) - \xi_f(0)}{\frac{1}{p^s}} \right| < \frac{\varepsilon}{p}.$$

Let  $\delta = \frac{1}{p^{s_0}}$ . If  $0 < x < \delta$ , then there exists  $s \in \mathbb{N}$  such that

$$\frac{1}{p^{s+1}} < x < \frac{1}{p^s} \leq \frac{1}{p^{s_0}}.$$

Therefore,

$$\begin{aligned} \left| \frac{\xi_f(x) - \xi_f(0)}{x} \right| &= \frac{\xi_f(0) - \xi_f(x)}{x} \\ &\leq \frac{\xi_f(0) - \xi_f\left(\frac{1}{p^s}\right)}{\frac{1}{p^{s+1}}} \\ &\leq p \times \frac{\varepsilon}{p} \\ &= \varepsilon \end{aligned}$$

as desired.  $\square$

Finally, we shall prove the last half of 3).

**Definition 2.8.** Let  $(S, \mathfrak{n})$  be a  $(d+1)$ -dimensional regular local ring. Let  $0 \neq \alpha \in \mathfrak{n}$ . The pair  $(\rho, \sigma)$  is called a *matrix factorization* of the element  $\alpha$  if all of the following conditions are satisfied:

- (1)  $\rho : G \rightarrow F$  and  $\sigma : F \rightarrow G$  are  $S$ -homomorphisms, where  $F$  and  $G$  are finitely generated  $A$ -free modules, and  $\text{rank}_S F = \text{rank}_S G$ .
- (2)  $\rho \circ \sigma = \alpha \cdot \text{id}_F$ .
- (3)  $\sigma \circ \rho = \alpha \cdot \text{id}_G$ .

Actually, if either (2) or (3) is satisfied, the other is satisfied.

**Definition 2.9.** Let  $(S, \mathfrak{n})$  be a  $(d+1)$ -dimensional regular local ring, and let  $0 \neq \alpha \in \mathfrak{n}$ . Let  $(\rho, \sigma)$  and  $(\rho', \sigma')$  be matrix factorizations of  $\alpha$ . We regard  $\rho$  and  $\sigma$  as  $r \times r$  matrixes with entries in  $S$ , and  $\rho'$  and  $\sigma'$  as  $r' \times r'$  matrixes with entries in  $S$ . Then, we write

$$(\rho, \sigma) \oplus (\rho', \sigma') = \left( \begin{pmatrix} \rho & 0 \\ 0 & \rho' \end{pmatrix}, \begin{pmatrix} \sigma & 0 \\ 0 & \sigma' \end{pmatrix} \right)$$

which is a matrix factorization of  $\alpha$ .

**Definition 2.10.** Let  $(S, \mathfrak{n})$  be a  $(d+1)$ -dimensional regular local ring, and let  $0 \neq \alpha \in \mathfrak{n}$ . A matrix factorization  $(\rho, \sigma)$  of  $\alpha$  is called *reduced* if all the entries of  $\rho$  and  $\sigma$  are in  $\mathfrak{n}$ .

**Remark 2.11.** Let  $(S, \mathfrak{n})$  be a  $(d+1)$ -dimensional regular local ring, and let  $0 \neq \alpha \in \mathfrak{n}$ . Let the map  $\alpha : S \rightarrow S$  be multiplication by  $\alpha \in \mathfrak{n}$  on  $S$ . If  $(\rho, \sigma)$  is a matrix factorization of  $\alpha \in \mathfrak{n}$ , then we can write

$$(\rho, \sigma) \simeq (\alpha, \text{id}_S)^{\oplus v} \oplus (\text{id}_S, \alpha)^{\oplus u} \oplus (\gamma_1, \gamma_2),$$

where  $v$  and  $u$  are some integers, and  $(\gamma_1, \gamma_2)$  is reduced. Therefore,

$$\begin{aligned} \text{cok}(\rho) &\simeq \text{cok}(\alpha)^{\oplus v} \oplus \text{cok}(id_S)^{\oplus u} \oplus \text{cok}(\gamma_1) \\ &\simeq (S/(\alpha))^{\oplus v} \oplus \text{cok}(\gamma_1). \end{aligned}$$

It is known that  $\text{cok}(\gamma_1)$  has no free direct summands if  $(\gamma_1, \gamma_2)$  is reduced ([4], Corollary 6.3). Consequently,  $v$  is equal to the largest rank of a free  $S/(\alpha)$ -module appearing as a direct summand of  $\text{cok}(\rho)$ .

Let  $F^e : R \rightarrow F_*^e R$  be the  $e$ -th Frobenius map. Consider the map  $f : F_*^e R \rightarrow F_*^e R$ . We have  $f = F_*^e(f^{p^e}) = F_*^e(f) \cdot F_*^e(f^{p^e-1}) = F_*^e(f^{p^e-1}) \cdot F_*^e(f)$ . Therefore,  $(F_*^e(f), F_*^e(f^{p^e-1}))$  is a matrix factorization. We put

$$(F_*^e(f), F_*^e(f^{p^e-1})) = (f, id_R)^{\oplus v_e} \oplus (id_R, f)^{\oplus u_e} \oplus (\text{reduced}).$$

By Remark 2.11 this implies that  $v_e$  is the number of  $R/(f)$  appearing as the direct summand of  $\frac{F_*^e R}{F_*^e(f)(F_*^e R)} = F_*^e(R/(f))$ . That is,  $\lim_{e \rightarrow \infty} \frac{v_e}{p^{en}}$  is the F-signature of  $R/(f)$ , denoted by  $s(R/(f))$ .

**Proposition 2.12.**  $v_e = \ell_R(M_{e, p^e-1})$ .

*Proof.* We can regard the map  $F_*^e(f^{p^e-1}) : F_*^e R \rightarrow F_*^e R$  as a  $p^{(n+1)e} \times p^{(n+1)e}$  matrix  $A$  with entries in  $R$ ;

$$A = \begin{pmatrix} I_{v_e} & & \\ & f & \\ & & \ddots \\ & & & f & \\ & & & & B \end{pmatrix},$$

where  $I_{v_e}$  is the identity matrix of size  $v_e$ , and  $B$  is a matrix with entries in  $\mathfrak{m}$ . Therefore, we have

$$v_e = \dim_{R/\mathfrak{m}} \left( \text{Im}(R/\mathfrak{m} \otimes F_*^e(f^{p^e-1})) \right) = \dim_{R/\mathfrak{m}} F_*^e \left( \frac{(f^{p^e-1}) + \mathfrak{m}^{[p^e]}}{\mathfrak{m}^{[p^e]}} \right) = M_{e, p^e-1}.$$

□

We completed a proof of Theorem 1.6.

**Remark 2.13.** Let  $(S, \mathfrak{n}, k)$  be a complete regular local ring of characteristic  $p > 0$ . Suppose that  $k$  is perfect. Let  $I$  be an ideal of  $S$ , and put  $\overline{S} = S/I$ . Suppose that  $a_e$  is equal to the largest rank of a free  $\overline{S}$ -module appearing in a direct summand of  $F_*^e \overline{S}$ . Then it is known that

$$a_e = \dim_k \frac{(I^{[p^e]} : I) + \mathfrak{m}^{[p^e]}}{\mathfrak{m}^{[p^e]}}$$

by Fedder's lemma (see [1]). If  $I = (f)$ , then

$$a_e = \dim_k \frac{(f^{p^e-1}) + \mathfrak{m}^{[p^e]}}{\mathfrak{m}^{[p^e]}}.$$

## 3. EXAMPLES

Let  $f = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{n+1}^{\alpha_{n+1}}$  and  $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+1}$ . We set  $(\underline{X}^{p^e}) = (X_1^{p^e}, X_2^{p^e}, \dots, X_{n+1}^{p^e})$  for  $e \geq 0$ . We have an exact sequence

$$0 \longrightarrow M_{e,t} \longrightarrow \frac{R}{(f^{t+1}) + (\underline{X}^{p^e})} \longrightarrow \frac{R}{(f^t) + (\underline{X}^{p^e})} \longrightarrow 0$$

for any  $t \geq 0$ . On the other hand, we have an exact sequence

$$0 \longrightarrow \frac{(f^t) + (\underline{X}^{p^e})}{(\underline{X}^{p^e})} \longrightarrow \frac{R}{(\underline{X}^{p^e})} \longrightarrow \frac{R}{(f^t) + (\underline{X}^{p^e})} \longrightarrow 0$$

for any  $t \geq 0$ . Hence,

$$\ell_R \left( \frac{R}{(f^t) + (\underline{X}^{p^e})} \right) = \begin{cases} p^{e(n+1)} - \prod_{j=1}^{n+1} (p^e - t\alpha_j) & (\text{if } t\alpha_{n+1} < p^e) \\ p^{e(n+1)} & (\text{otherwise}) \end{cases}$$

holds. Therefore, we have

$$\ell_R(M_{e,t}) = \begin{cases} 0 & \left( \frac{p^e}{\alpha_{n+1}} \leq t \right) \\ \prod_{j=1}^{n+1} (p^e - t\alpha_j) & \left( \frac{p^e}{\alpha_{n+1}} - 1 \leq t < \frac{p^e}{\alpha_{n+1}} \right) \\ \prod_{j=1}^{n+1} (p^e - t\alpha_j) - \prod_{j=1}^{n+1} (p^e - (t+1)\alpha_j) & \left( t < \frac{p^e}{\alpha_{n+1}} - 1 \right) \end{cases} \quad (10)$$

If  $t < \frac{p^e}{\alpha_{n+1}} - 1$ ,

$$\begin{aligned} \ell_R(M_{e,t}) &= \prod_{j=1}^{n+1} (p^e - t\alpha_j) - \prod_{j=1}^{n+1} (p^e - (t+1)\alpha_j) \\ &= \sum_{j=1}^{n+1} (-1)^j t^j \beta_j p^{e(n+1-j)} - \sum_{j=1}^{n+1} (-1)^j (t+1)^j \beta_j p^{e(n+1-j)} \\ &= \sum_{j=1}^{n+1} (-1)^{j+1} \left( \sum_{i=0}^{j-1} \binom{j}{i} t^i \right) \beta_j p^{e(n+1-j)}, \end{aligned}$$

where  $\beta_j$  denotes the elementary symmetric polynomial of degree  $j$  in  $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ . Hence

$$C_{e,t} = \frac{\ell_R(M_{e,t})}{p^{en}} = \sum_{j=1}^{n+1} (-1)^{j+1} \left( \sum_{i=0}^{j-1} \binom{j}{i} \frac{t^i}{p^{e(j-1)}} \right) \beta_j$$

holds. We shall calculate  $\xi_f(x)$ . If  $x < \frac{1}{\alpha_{n+1}}$ , then  $\lfloor xp^e \rfloor < \frac{p^e}{\alpha_{n+1}} - 1$  for  $e \gg 0$ .

Then,

$$C_{e, \lfloor xp^e \rfloor} = \sum_{j=1}^{n+1} (-1)^{j+1} \left( \sum_{i=0}^{j-1} \binom{j}{i} \frac{\lfloor xp^e \rfloor^i}{p^{e(j-1)}} \right) \beta_j.$$

Since  $x p^e - 1 \leq \lfloor x p^e \rfloor \leq x p^e$ , we have

$$\lim_{e \rightarrow \infty} \frac{\lfloor x p^e \rfloor^a}{p^{eb}} = \begin{cases} x^a & (\text{if } a = b) \\ 0 & (\text{if } a < b) \end{cases}.$$

Consequently,

$$\xi_f(x) = \beta_1 - 2\beta_2 x + 3\beta_3 x^2 - \cdots + (-1)^n (n+1) \beta_{n+1} x^n \quad (11)$$

holds for  $0 \leq x < \frac{1}{\alpha_{n+1}}$ . In particular,  $e_{HK}(R/(f)) = \xi_f(0) = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}$ .

By (10), we have

$$\xi_f(x) = 0 \quad (12)$$

if  $x > \frac{1}{\alpha_{n+1}}$ .

Next, we shall calculate  $\xi_f\left(\frac{1}{\alpha_{n+1}}\right)$ . Since  $\frac{p^e}{\alpha_{n+1}} - 1 \leq \left\lfloor \frac{p^e}{\alpha_{n+1}} \right\rfloor \leq \frac{p^e}{\alpha_{n+1}}$  for any  $e \geq 0$ ,

$$\begin{aligned} \ell(M_{e, \lfloor \frac{1}{\alpha_{n+1}} p^e \rfloor}) &= \prod_{j=1}^{n+1} \left\{ p^e - \left\lfloor \frac{p^e}{\alpha_{n+1}} \right\rfloor \alpha_j \right\} \\ &= \varepsilon_e \prod_{j=1}^n \left\{ p^e - (p^e - \varepsilon_e) \frac{\alpha_j}{\alpha_{n+1}} \right\} \\ &= \varepsilon_e \prod_{j=1}^n \left\{ \left( 1 - \frac{\alpha_j}{\alpha_{n+1}} \right) p^e + \frac{\varepsilon_e}{\alpha_{n+1}} \alpha_j \right\} \\ &= \varepsilon_e \left( \frac{1}{\alpha_{n+1}} \right)^n \prod_{j=1}^n \{ (\alpha_{n+1} - \alpha_j) p^e + \varepsilon_e \alpha_j \} \\ &= \varepsilon_e \left( \frac{1}{\alpha_{n+1}} \right)^n \left\{ p^{en} \prod_{j=1}^n (\alpha_{n+1} - \alpha_j) + \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \delta_{\underline{i}} p^{e(n-k)} \varepsilon_e^k \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \right\} \\ &= \varepsilon_e \left( \frac{1}{\alpha_{n+1}} \right)^n p^{en} \left\{ \prod_{j=1}^n (\alpha_{n+1} - \alpha_j) + \sum_{k=1}^n \delta_k \left( \frac{\varepsilon_e}{p^e} \right)^k \right\}, \end{aligned}$$

where  $\varepsilon_e \equiv p^e \pmod{\alpha_{n+1}}$  such that  $0 \leq \varepsilon_e < \alpha_{n+1}$ , and

$$\begin{aligned} \delta_{\underline{i}} &= \prod_{j \neq i_1, i_2, \dots, i_k} (\alpha_{n+1} - \alpha_j), \\ \delta_k &= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \delta_{\underline{i}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}. \end{aligned}$$

Hence,

$$C_{e, \lfloor \frac{1}{\alpha_{n+1}} p^e \rfloor} = \varepsilon_e \left( \frac{1}{\alpha_{n+1}} \right)^n \left\{ \prod_{j=1}^n (\alpha_{n+1} - \alpha_j) + \sum_{k=1}^n \delta_k \left( \frac{\varepsilon_e}{p^e} \right)^k \right\},$$

and therefore

$$\limsup_{e \rightarrow \infty} C_{e, \lfloor \frac{1}{\alpha_{n+1}} p^e \rfloor} = \left( \limsup_{e \rightarrow \infty} \varepsilon_e \right) \left( \frac{1}{\alpha_{n+1}} \right)^n \prod_{j=1}^n (\alpha_{n+1} - \alpha_j). \quad (13)$$

We shall examine whether  $\lim_{e \rightarrow \infty} \varepsilon_e$  exists. Let  $\alpha_{n+1} = p^s q$ , where  $q$  is coprime to  $p$ , and  $s$  is a non-negative integer. If  $p \equiv 1 \pmod{q}$ , then we can find that  $\varepsilon_e$  is constant for any  $e \geq s$  by the Chinese remainder theorem. If  $p \not\equiv 1 \pmod{q}$ , then  $\varepsilon_e$  is eventually periodic with period more than 1.

From the following Proposition 3.1, we get to know the function  $\xi_f(x)$ .

**Proposition 3.1.** Let  $f = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{n+1}^{\alpha_{n+1}}$  with  $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+1}$ .

- 1) We have  $\text{fpt}(f) = \frac{1}{\alpha_{n+1}}$ . If  $\alpha_{n+1} \geq 2$ , we have  $s(R/(f)) = 0$ .
- 2)  $\lim_{x \rightarrow \frac{1}{\alpha_{n+1}} - 0} \xi_f(x) = \left( \frac{1}{\alpha_{n+1}} \right)^{n-1} \prod_{j=1}^n (\alpha_{n+1} - \alpha_j) \geq 0$ .
- 3) The function  $\xi_f(x)$  is continuous on  $[0, 1]$  if and only if  $\alpha_{n+1} = \alpha_n$  holds.
- 4) Let  $\alpha_{n+1} = p^s q$ , where  $q$  is coprime to  $p$ , and  $s$  is a non-negative integer. The limit  $\lim_{e \rightarrow \infty} \xi_{f,e} \left( \frac{1}{\alpha_{n+1}} \right)$  exists if and only if it satisfies that  $\alpha_{n+1} = \alpha_n$  or  $p \equiv 1 \pmod{q}$ .

*Proof.* By (11) and (12), we obtain 1) immediately.

Next we shall prove 2). We set

$$g(x) = \beta_1 - 2\beta_2 x + 3\beta_3 x^2 - \cdots + (-1)^n (n+1) \beta_{n+1} x^n$$

and

$$h(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{n+1}).$$

Now, since  $h(x) = x^{n+1} - \beta_1 x^n + \beta_2 x^{n-1} - \cdots + (-1)^{n+1} \beta_{n+1}$ ,

$$x^{n+1} h \left( \frac{1}{x} \right) = 1 - \beta_1 x + \beta_2 x^2 - \cdots + (-1)^{n+1} \beta_{n+1} x^{n+1}.$$

Hence, we have the following equation

$$g(x) = - \left\{ x^{n+1} h \left( \frac{1}{x} \right) \right\}' = -(n+1) x^n h \left( \frac{1}{x} \right) + x^{n-1} h' \left( \frac{1}{x} \right).$$

Since  $h(\alpha_{n+1}) = 0$ ,

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{\alpha_{n+1}} - 0} \xi_f(x) &= g \left( \frac{1}{\alpha_{n+1}} \right) \\ &= \left( \frac{1}{\alpha_{n+1}} \right)^{n-1} h'(\alpha_{n+1}) \\ &= \left( \frac{1}{\alpha_{n+1}} \right)^{n-1} \prod_{j=1}^n (\alpha_{n+1} - \alpha_j) \geq 0. \end{aligned}$$

The assertion 3) follows from (10), (11) and 2) as above. The assertion 4) follows from the equation (13).  $\square$

**Example 3.2.** If  $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-2} = 0$  and  $\alpha_{n-1} \neq 0$ , the derivative

$$g'(x) = -2(\alpha_{n+1}\alpha_n + \alpha_{n+1}\alpha_{n-1} + \alpha_n\alpha_{n-1}) + 6\alpha_{n+1}\alpha_n\alpha_{n-1}x.$$

Let  $\alpha$  be the root of  $g'(x) = 0$ , that is,

$$\alpha = \frac{1}{3} \times \frac{\alpha_{n+1}\alpha_n + \alpha_{n+1}\alpha_{n-1} + \alpha_n\alpha_{n-1}}{\alpha_{n+1}\alpha_n\alpha_{n-1}}.$$

Then, we have

$$\alpha - \frac{1}{\alpha_{n+1}} = \frac{1}{\alpha_{n+1}} \left\{ \frac{1}{3} \left( \frac{\alpha_{n+1}}{\alpha_{n-1}} + \frac{\alpha_{n+1}}{\alpha_n} + 1 \right) - 1 \right\} \geq 0,$$

and so  $g'(x) < 0$  for any  $x < \frac{1}{\alpha_{n+1}}$ . Moreover, if  $\alpha_{n+1} \neq \alpha_n$  we obtain  $g' \left( \frac{1}{\alpha_{n+1}} \right) < 0$ . The second derivative  $g''(x)$  is positive for any  $x \in \mathbb{R}$ . In fact,  $g''(x) = 6\alpha_{n-1}\alpha_n\alpha_{n+1} > 0$ .

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